Mathematics Worksheet for
Atmosphere, Ocean and Climate MSc

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This worksheet includes revision notes and exercises on some aspects of complex numbers, algebra, calculus, trigonometry and curve sketching. This mathematics is useful in science degrees. If you are doing the Atmosphere, Ocean and Climate (AOC) MSc then you should attempt this worksheet before the beginning of term. You can ask questions about it when you start and you will be asked to hand it in near the beginning of term. You can write answers on this question sheet where there is space or otherwise on separate sheets. Don't worry if you cannot do all of the worksheet. This is not assessed. It will help the staff identify areas of mathematics where you will need more help. The answers will be available after the hand-in date.

1 Curve Sketching

Make rough sketches of the following curves, marking on locations of crossing axes, turning point values and locations and any asymptotic limits. You should be able to do this without evaluating multiple points. There are a lot of them because they build in complexity and doing them in order will help to do the more difficult ones towards the end.

1. $y = x^2$
2. $y = 2x^2$
3. $y = x^2 + 1$
4. $y = (x-1)^2$
5. $y = x(1-x)$
6. $y = 1/x$
7. $y = 1/(1-x)$
8. $y = \sqrt{1+x^2}$
9. $y = \sqrt{1-x^2}$
10. $y = (x + |x|)/(1 + |x|)$
11. $y = \sin x$
12. $y = \cos 2x$
13. $y = \tan \dfrac{\pi x}{2}$
14. $y = \tan^{-1} x$
15. $y = 1 - \cos 2x$
16. $y = 1/(1-\cos 2x)$
17. $y = \sin^2 x$
18. $y = \sin x + \dfrac{1}{4}\sin 10x$
19. $y = e^{-x}$
20. $y = -e^{-x}$
21. $y = e^{-x}\cos 2\pi x$

Later in the term you will be able to check your curves by plotting them with Python.

2 Matrices and Algebra

1. Solve the following systems of simultaneous equations:

(a)
$$\begin{aligned} 2x + 3y &= 13 \\ 5x - y &= 7 \end{aligned}$$

(b)
$$\begin{aligned} 3x + y + 2z &= 3 \\ -x - y + z &= 7 \\ 2x + 2y + z &= -2 \end{aligned}$$

(c) Assuming that e, f, x, and y are known, find n:

$$\begin{aligned} e &= ax^n \\ f &= ay^n \end{aligned}$$

(d) Rearrange to give y as a function of the others. You should be able to express your answer without any logarithms.
$x = a \ln y + c \ln z$

2. Express the simultaneous equations in questions 1a and 1b above as matrix equations.

3. Rearrange to find A as a function of c:

   (a) $A = \frac{1}{A} + 2c$

   (b) $A = 1 + cA$

4. Which of the following matrices can be inverted?
   (a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, (c) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, (e) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

2.1 Equating coefficients

If we know that an expression should hold for EVERY value of x, then the same expression should hold for each power of x separately. For example if we know that:

$$a + bx + cx^2 = 1 + 2x - 3x^2$$

and we know that this expression holds for ALL values of x then we can equate coefficients of x:

Equating coefficients of x^0: $a = 1$
Equating coefficients of x^1: $b = 2$
Equating coefficients of x^2: $c = -3$

2.1.1 Question

If we know that the following expression is true for all values of c, find Ψ as a function of the ϕs:

$$\tfrac{1}{2}(3-c)\phi_j - \tfrac{1}{2}(1-c)\phi_{j-1} = \tfrac{1}{2}\Psi\left((1+c)\phi_j + (1-c)\phi_{j+1}\right) + (1-\Psi)\phi_j$$

3 Surds

Reference: http://www.bbc.co.uk/schools/gcsebitesize/maths/number/surdsrev1.shtml

A surd is a square root which cannot be reduced to a whole number. For example, $\sqrt{4} = 2$ is not a surd, as the answer is a whole number. But $\sqrt{5}$ is not a whole number. You could use a calculator to find that $\sqrt{5} = 2.236067977$ but instead of this we often leave our answers in the square root form, as a surd.

You need to be able to simplify expressions involving surds. Here are some rules that you will need to learn:

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$
$$\sqrt{a}\sqrt{a} = a.$$

You will also need to be able to rationalise expressions including surds. This means removing any square roots from the denominator. This is done by multiplying the top and bottom of the fraction by the necessary expression. For example:

$$\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}}\frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}$$

and

$$\frac{f}{a-\sqrt{b}} = \frac{f}{a-\sqrt{b}}\frac{a+\sqrt{b}}{a+\sqrt{b}} = \frac{f\left(a+\sqrt{b}\right)}{a^2 + a\sqrt{b} - a\sqrt{b} - b^2} = \frac{f\left(a+\sqrt{b}\right)}{a^2 - b^2}$$

Without using a calculator, simplify the following expressions and express using surds:

1. $\sqrt{12}$
2. $\sqrt{12}\sqrt{3}$
3. $\dfrac{\sqrt{12}}{\sqrt{6}}$
4. $\dfrac{\sqrt{8}}{\sqrt{6}}$
5. $\dfrac{1-\sqrt{2}}{1+\sqrt{2}}$

4 Trigonometry

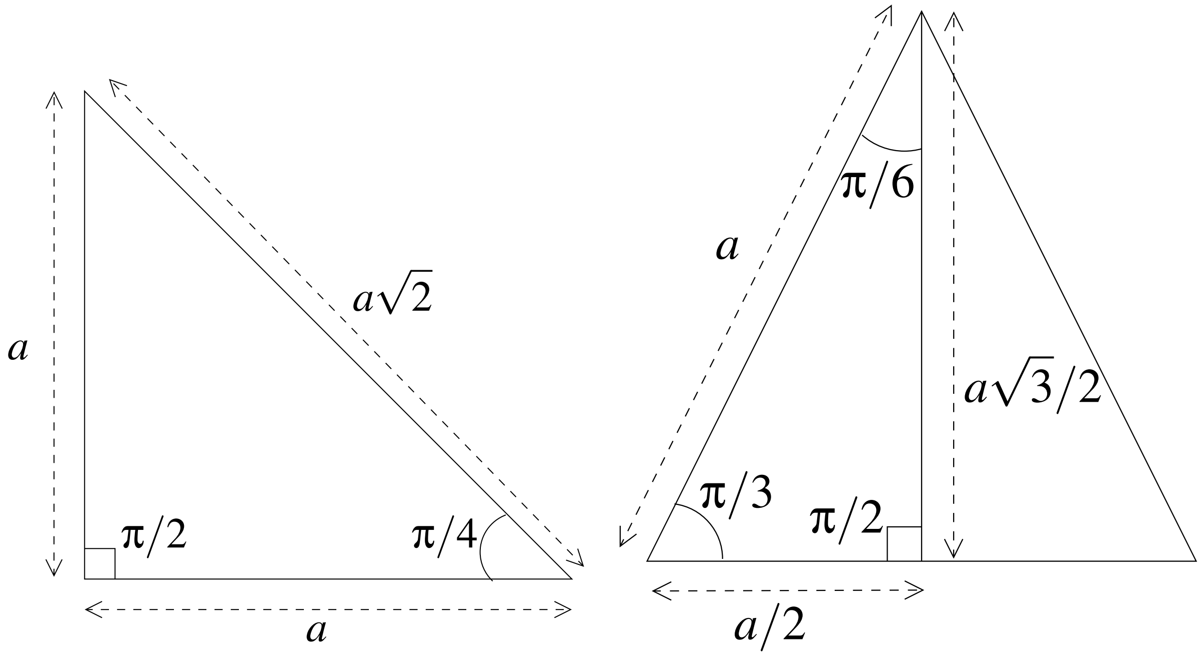

Figure 1: Angles and sides of equilateral and right-angle triangles

Using figure 1, express the following as fractions including surds and multiples of π (assume that angles are in radians). For inverses, make sure that you give all answers.

1. $\sin(\pi/6)$
2. $\cos(\pi/6)$
3. $\tan(\pi/6)$
4. $\sin(\pi/4)$
5. $\cos(\pi/4)$
6. $\tan(\pi/4)$
7. $\sin(\pi/3)$
8. $\cos(\pi/3)$
9. $\tan(\pi/3)$
10. $\sin(\pi/2)$
11. $\cos(\pi/2)$
12. $\tan(\pi/2)$
13. $\sin(2\pi/3)$
14. $\cos(3\pi/4)$
15. $\tan\pi$
16. $\sin(4\pi/3)$
17. $\cos(3\pi/2)$
18. $\tan(7\pi/4)$
19. $\sin^{-1} 0$
20. $\cos^{-1} 0$
21. $\tan^{-1} 0$
22. $\sin^{-1} 1$
23. $\cos^{-1}(1/2)$
24. $\tan^{-1} 1$

4.1 Trigonometric Identities

The Pythagorean identity is

$$\sin^2\theta + \cos^2\theta = 1$$

and the double angle formulae for $\sin$ and $\cos$ are:

$$\begin{aligned}\sin(\alpha\pm\beta) &= \sin\alpha\cos\beta \pm \sin\beta\cos\alpha \\ \cos(\alpha\pm\beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta\end{aligned}$$

Let us define $s = \sin\theta$ and $c = \cos\theta$. Express the following in terms solely of s, c, π and θ, without any trigonometric functions:

1. $\tan(\theta)$
2. $\sin(2\theta)$
3. $\cos(2\theta)$ (give three alternatives, one in terms of s and c, one in terms only of c and the other in terms only of s)
4. $\sin(\pi/2-\theta)$
5. $\cos(\pi/2-\theta)$
6. $\sin^{-1}(\cos(\theta))$

5 Complex numbers

Complex numbers are used extensively in numerical analysis but are not on all mathematics A-level syllabuses. Here therefore is a quick crash course (or revision).

5.1 The Argand Diagram and Complex number arithmetic

Imaginary numbers are square roots of real, negative numbers (ie they do not exist, but we can imagine them). The first imaginary number is:

$$i=\sqrt{-1}$$

All other imaginary numbers are multiples of i. They therefore lie on a number-line, like the real numbers. These two number-lines can be drawn at right angles to each other to create the Argand diagram (figure 2).

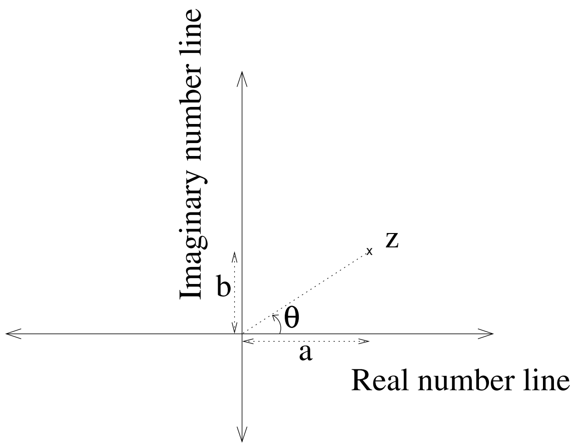

Figure 2: A complex number, z, represented on the real and imaginary axes of the Argand diagram.

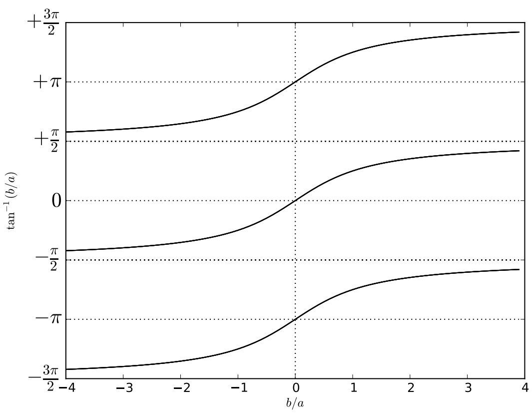

Figure 3: The multiple values of $\tan^{-1}$

A complex number is expressed as the sum of a real and imaginary number:

$$z=a+ib$$

where a and b are real. This can be represented on the Argand diagram as shown in figure 2. The magnitude, r, of the complex number is the distance to zero:

$$r=\sqrt{a^2+b^2}$$

and the argument is the angle, θ, between the complex number and the positive real axis.

$$\theta=\tan^{-1}\frac{b}{a}.$$

Remember that $\tan^{-1}$ is a multi-valued function (fig 3) so you must take care to find the correct value by considering the signs of a and b independently. The representation of z as (r,θ) is described as polar form.

Complex numbers can be added and subtracted by adding and subtracting their real and imaginary parts separately.

5.1.1 Excercises

Calculate:
$i^2 =$ $\quad (2i)^2 =$ $\quad i^3 =$ $\quad i^4 =$

If $z_1 = 2 + 5i$ and $z_2 = -1 + i$, what are:

1. $z_1 + z_2$
2. $z_1 - z_2$
3. The magnitude of z_1 and z_2
4. The arguments of z_1 and z_2

5.1.2 Multiplication and Division

Complex numbers are multiplied by multiplying out the brackets and replacing and i^2 terms with -1:

$$\begin{aligned}(2+5i)(-1+i) &= -2 + 2i - 5i + 5i^2 \\ &= -2 - 3i - 5 \\ &= -7 - 3i\end{aligned}$$

In order to divide by a complex number it is necessary to multiply top and bottom by the complex conjugate:

For complex number $z = a + ib$, the complex conjugate is $z^* = a - ib$

Hence:

$$\begin{aligned}\frac{2+5i}{-1+i} &= \frac{(2+5i)(-1-i)}{(-1+i)(-1-i)} \\ &= \frac{3-7i}{2} = \frac{3}{2} - \frac{7}{2}i\end{aligned}$$

A complex number multiplied by its complex conjugate is always equal to the magnitude squared:

$$zz* = |z|^2$$

5.1.3 Excercises

Calculate the following, assuming that a, b, c and d are real numbers:

1. $(a + ib)(c + id) =$
2. $\dfrac{a + ib}{c + id} =$

5.2 Roots of quadratic equations

Complex numbers can be used to express roots of quadratic equations that do not have real roots. The quadratic equation:

$$ax^2 + bx + c = 0$$

has roots

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If the discriminant, $b^2 - 4ac$, is less that zero, then the roots are complex:

$$x = -\frac{b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a}.$$

5.2.1 Excercise

Write down the roots of the quadratic equation $x^2 - 2x + 3 = 0$

5.3 Euler's Identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

If a complex number in polar form is (r, θ), then it can be expressed as a real and imaginary part using Euler's identity:

$$z = re^{i\theta} = r\cos\theta + ir\sin\theta$$

Multiplication and division are more straightforward using polar representation:

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)) \end{aligned}$$

5.3.1 Exercise

Calculate the following:

1. $5e^{i\frac{\pi}{7}} \times 2e^{i\frac{\pi}{42}} =$
2. $e^{i\pi} =$
3. $\dfrac{4e^{i\frac{\pi}{3}}}{2e^{i\frac{\pi}{6}}} =$
4. Using your knowledge of division of complex numbers in polar form, calculate $|A|^2$ given $A = \dfrac{a+ib}{c+id}$ without multiplying the top and bottom by the complex conjugate of the denominator.

   Simplify the following

5. $1/2\left(e^{i\theta} + e^{-i\theta}\right)$
6. $1/2\left(e^{i\theta} - e^{-i\theta}\right)$
7. We will consider what happens to the real part of an oscillation with wavenumber k when it is multiplied by a single complex number. First sketch the real part of $e^{i\theta}$. Next multiply $e^{i\theta}$ by $a + ib = re^{i\alpha}$ and express the answer in polar form. On the same graph, sketch the real part of the solution (assuming that $1 < r < 2$ and that $0 < \alpha << \pi/k$). Hence describe the influence of both r and α on the wave.

These will prove very useful in some of the numerical analysis later in the course.

5.4 Complex Numbers in Differential Equations

5.4.1 Exercise

If the differential equation	$\dfrac{d\phi}{dt} = -\lambda\phi$ with $\phi(t=0) = \phi_0$ and $\lambda \in \mathbb{R}$
has the solution	$\phi = \phi_0 e^{-\lambda t}$
what is the solution of the differential equation:	$\dfrac{d\phi}{dt} = -i\omega\phi$ with $\phi(t=0) = \phi_0$ with $\omega \in \mathbb{R}$

What is the behaviour of ϕ for each of these two situations with $\lambda > 0$ and with $\omega > 0$?

6 Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors are used extensively in numerical analysis and in data analysis including analysing meteorological data.

Any $N \times N$ matrix, A, has N eigenvalues, λ, and N eigenvectors, $\mathbf{v}$, such that:

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{1}$$

The eigenvalues and eigenvectors may be real or complex valued and they may not all be unique. The eigenvalues can be found by solving the characteristic equation:

$$\det(A - \lambda I) = 0 \tag{2}$$

where I is the identity matrix and det is the matrix determinant. Equation (2) is an N^{th} order polynomial so it has N roots which are the N eigenvalues. Once the eigenvalues are found, each eigenvector can be found by solving

$$(A - \lambda I)\,\mathbf{v} = 0. \tag{3}$$

6.1 Example

Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$.

Solve $\det\left(\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$.

$\det\left(\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\left(\begin{pmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{pmatrix}\right) = \lambda(3+\lambda) + 2 = 0.$

$\implies \lambda^2 + 3\lambda + 2 = 0 \implies \lambda = \frac{1}{2}(-3 \pm 1) = -2, -1.$

6.2 Exercise

Find the eigenvalues of the matrices:

1. $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$
2. $A = \begin{pmatrix} 7 & 3 \\ 3 & -1 \end{pmatrix}$

7 Div, grad and curl

In three dimensions, there are a few different possibilities for taking differentials of scalar and vector fields. The div operator $(\nabla\cdot)$ evaluates the divergence of a vector field. The grad operator (∇) evaluates the (vector valued) gradient of a scalar field. The curl operator $(\nabla\times)$ evaluates the curvature of a vector field. Given a vector field, $\mathbf{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ (vector components in the x, y and z directions), a scalar field, ϕ, both with variation in the x, y and z directions, div, grad and curl are defined:

$$\mathrm{div}\mathbf{u} = \nabla\cdot\mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad \mathrm{grad}\phi = \nabla\phi = \begin{pmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{pmatrix} \quad \mathrm{curl}\mathbf{u} = \nabla\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \begin{pmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix}.$$

There is also the Laplacian: $\nabla^2\phi = \nabla\cdot(\nabla\phi)$.

Given these definitions and any vector calculus identities that you might be able to find, answer the following questions:

1. Express the equation $\nabla^2\phi$ in component form (in terms of variations in x,y and z)
2. Express the vector equation:

$$\frac{\partial \mathbf{u}}{\partial t} = 2\Omega \times \mathbf{u} - \nabla\phi$$

   in component form (in terms of u,v,w,x,y and z) where $\mathbf{u} = \begin{pmatrix} u \\ v \\ 0 \end{pmatrix}$, $\Omega = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$ and ϕ varies only in the x and y directions.

3. Express the equation $\nabla \times \nabla \phi$ in component form.
4. Derive the following chain rule (by writing in component form):

$$\nabla \cdot (\phi \mathbf{u}) = \phi \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \phi$$

5. Given a divergence free vector field ($\nabla \cdot \mathbf{u} = 0$), use the chain rule above to find an alternative form for the equation:

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{u}) = 0$$

7.1 Gauss's Divergence Theorem

Reference: http://mathworld.wolfram.com/DivergenceTheorem.html

If V is a region in space with boundary S then the volume integral of the divergence $\nabla \cdot \mathbf{F}$ where $\mathbf{F}$ is a non-uniform vector field and the surface integral of $\mathbf{F}$ over the boundary S of V are related by

$$\int_V \nabla \cdot \mathbf{F} dV = \int_S \mathbf{F} \cdot \mathbf{dS}$$

where $\mathbf{dS}$ is the outward pointing normal vector to the surface S whose size is proportional to the area. Gauss's divergence theorem is a mathematical statement of the physical fact that, in the absence of the creation or destruction of matter, the density within a region of space can change only by having it flow into or away from the region through its boundary.

7.1.1 Exercise

Imagine a cube shaped region of space with side length 10m. The table below gives the centres of each of the cube faces and the average wind velocity over each face and the unit normal vector for each face

Face centre	Average wind velocity (ms^{-1})	Unit normal vector
$(5,5,0)$	$(0,0,0)$	$(0,0,-1)$
$(5,5,10)$	$(1,1,0)$	$(0,0,1)$
$(0,5,5)$	$(1,1,0)$	$(-1,0,0)$
$(10,5,5)$	$(0,1,0)$	$(1,0,0)$
$(5,0,5)$	$(1,1,0)$	$(0,-1,0)$
$(5,10,5)$	$(0,1,0)$	$(0,1,0)$

Evaluate the average wind divergence in the box, $\frac{1}{V}\int_V \nabla \cdot \mathbf{u} dV$ using Gauss's divergence theorem and give units for your answer.